Online Appendix for "Fragile Stable Matchings"

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Abstract

This Online Appendix contains the following items. First, in Appendix A, we discuss basic properties of fragments that the main text refers to. Second, in Appendix B, we provide all proofs omitted from the main text. Third, Appendix C includes simulation results for dynamics where, at each step, a randomly-chosen agent selects her most preferred blocking partner. Fourth, in Appendix D, we present results and techniques related to probabilistic aspects of fragments. Fifth, Appendix E examines additional classes of natural dynamics beyond those discussed in the paper. Sixth, Appendix F presents two examples, mentioned in the main text, that quantify and compare the fragility of different stable matchings.

A. Properties of Fragments

In this section, we provide more details behind the properties of fragments and induced matchings stated in Section 3.1 of the main text.

Example 3. *The example shows that a fragment may be induced by multiple matchings.*

Consider the following market with three firms and three workers:

$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{pmatrix} 3,2 & 2,3 & 1,1 \\ 2,3 & 3,2 & 1,2 \\ 3,1 & 2,1 & 1,3 \end{pmatrix} \end{array}.$$

Firms $\bar{F} = \{f_1, f_2\}$ and workers $\bar{W} = \{w_1, w_2\}$ form a fragment that can be induced by either

$$\bar{\mu}_1 = \begin{pmatrix} w_1 & w_2 \\ | & | \\ f_1, & f_2 \end{pmatrix} \quad \text{or} \quad \bar{\mu}_2 = \begin{pmatrix} w_1 & w_2 \\ | & | \\ f_2, & f_1 \end{pmatrix}. \qquad \triangle$$

Lemma 4 shows that for any matching inducing a fragment, there exists a stable matching in the original market that coincides with the inducing matching over the fragment.

Lemma 4. *Consider any matching $\bar{\mu}$ that induces fragment $(\bar{F}, \bar{W})$. Then, there exists a stable matching μ for the original market that agrees with $\bar{\mu}$ when restricted to $\bar{F} \cup \bar{W}$.*[1]

Proof. The submarket obtained from the original market by removing the given fragment has a stable matching. Any such stable matching merged with the inducing matching constitutes a stable matching in the original market, as desired. ∎

Nonetheless, a stable matching in the original market may disagree over a fragment with all matchings that induce the fragment. Indeed, in Example 2 from the main text, the worker-optimal stable matching μ_W disagrees with the unique inducing matching $\bar{\mu}$.

Section 3.1 of the main text shows that a sequence of top-top match pairs forms a trivial fragment. Example 4 presents a trivial fragment that does not correspond to such a sequence.

[1]This lemma implies that only "projections" of stable matchings in the original market might potentially induce fragments.

Example 4. *The example shows that trivial fragments are not limited to sequences of top-top match pairs.*

Consider the following market with four firms and four workers:

$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{array}\begin{array}{c} \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{pmatrix} 2,4 & 1,4 & 3,2 & \mathbf{4,3} \\ 3,3 & 2,1 & 4,3 & 1,1 \\ 4,1 & 3,2 & 1,4 & 2,2 \\ 2,2 & \mathbf{4,3} & 3,1 & 1,4 \end{pmatrix} \end{array}.$$

It has two stable matchings:

$$\mu_F = (f_3, f_4, f_2, f_1) \quad \text{and} \quad \mu_W = (f_2, f_4, f_3, f_1).$$

Since both stable matchings agree with the matching

$$\bar{\mu} = \begin{pmatrix} w_2 & w_4 \\ | & | \\ f_4, & f_1 \end{pmatrix}$$

that induces fragment $(\bar{F}, \bar{W}) = (\{f_1, f_4\}, \{w_2, w_4\})$, the considered fragment is trivial. However, there are no top-top matches in this market. △

B. Omitted Proofs

Lemma 5. *Consider the random walk $\{S_i\}_{i\geq 0}$ defined in the proof of Proposition 1 from the main text. For small enough $\zeta > 0$, the number of steps this random walk takes to first reach $S_i \geq n$ is $2^{\Omega(n)}$ with probability $1 - 2^{-\Omega(n)}$.*

Proof. The proof of this lemma follows from standard Chernoff bound arguments and is similar to Ackermann et al. (2011).

Multiplicative Chernoff Bound (e.g., Theorem 4.1 in Motwani and Raghavan, 1995)**.** *Let $\{X_i\}_{i\in[n]}$ be independent Poisson trials such that, for $i \in [n]$, $\Pr[X_i = 1] = p_i$, where $0 < p_i < 1$. Then, for $X = \sum_{i=1}^{n} X_i$ and any $\gamma > 0$,*

$$\Pr\left[X \geq (1+\gamma)\,\mathbb{E}[X]\right] \leq \left[\frac{e^{\gamma}}{(1+\gamma)^{(1+\gamma)}}\right]^{\mathbb{E}[X]}.$$

For simplicity, consider another random walk $\{T_i\}_{i\geq 0}$, on the set $\{0, 1, 2, \ldots\}$, which we describe below. This new random walk starts at $T_0 = 0$. When $T_i = 0$ for some $i \geq 0$, including starting $i = 0$, the random walk deterministically moves four units to the right,

$T_{i+1} = T_i + 4$. If instead $T_i > 0$, the random walk moves one unit to the left, $T_{i+1} = T_i - 1$, with probability

$$p_{d^\star} = \frac{\eta - \zeta}{\eta + (2\kappa - 1)\zeta} \to 1 \text{ as } \zeta \to 0,$$

and four units to the right, $T_{i+1} = T_i + 4$, with the remaining probability $p_{s^\star} = 1 - p_{d^\star} \to 0$ as $\zeta \to 0$. The number of steps it takes for the original random walk $\{S_i\}_{i \geq 0}$ to first reach $S_i \geq n$ is identical to the number of steps it takes for the new random walk $\{T_i\}_{i \geq 0}$ to first reach $T_i \geq n - \lceil (1-\zeta)n \rceil$. Henceforth, we examine the latter number.

If the random walk is at position $T_i \geq n - \lceil (1-\zeta)n \rceil$ after i steps, then for some $j \leq i - (n - \lceil (1-\zeta)n \rceil)/4$, the following event $\mathcal{F}_{i,j}$ must occur:

$$T_i \geq n - \lceil (1-\zeta)n \rceil, \quad T_j = 0, \quad \text{and} \qquad \forall k \in \{j+1, j+2, \ldots, i-1\}, \quad T_k > 0.$$

Let $R_{i,j}$ denote the number of times the random walk increases its position by four and $L_{i,j}$ denote the number of times the random walk decreases its position by one during the steps $j+1, j+2, \ldots, i$. When the event $\mathcal{F}_{i,j}$ occurs, we must have $4R_{i,j} - L_{i,j} \geq n - \lceil (1-\zeta)n \rceil$. This inequality is equivalent to $5R_{i,j} \geq n - \lceil (1-\zeta)n \rceil + i - j$ since $L_{i,j} + R_{i,j} = i - j$. This, in turn, implies that

$$R_{i,j} \geq \frac{i-j}{5} = \frac{\eta + (2\kappa - 1)\zeta}{10\kappa\zeta} \times \mathbb{E}[R_{i,j}] \quad \text{since} \quad \mathbb{E}[R_{i,j}] = p_{s^\star} \times (i-j) = \frac{2\kappa\zeta}{\eta + (2\kappa-1)\zeta} \times (i-j).$$

Therefore,

$$\Pr[\mathcal{F}_{i,j}] \leq \Pr\left[R_{i,j} \geq (1+\gamma) \times \mathbb{E}[R_{i,j}]\right], \quad \text{where} \quad \gamma \equiv \frac{\eta - (8\kappa + 1)\zeta}{10\kappa\zeta}.$$

Furthermore, for small enough $\zeta > 0$, we must have large enough $\gamma > 0$, so that

$$\frac{e^\gamma}{(1+\gamma)^{(1+\gamma)}} < 1.$$

In addition, $\mathbb{E}[R_{i,j}] = \Omega(n)$. Indeed, $\mathbb{E}[R_{i,j}] = p_{s^\star} \times (i-j)$ and the event $\mathcal{F}_{i,j}$ can only occur if $i - j = \Omega(n)$. Consequently, by the multiplicative Chernoff bound,

$$\Pr[\mathcal{F}_{i,j}] \leq \Pr\left[R_{i,j} \geq (1+\gamma) \times \mathbb{E}[R_{i,j}]\right] \leq 2^{-cn} \quad \text{for some} \quad c > 0.$$

Finally, we apply the standard union bound trick to upper-bound the probability that the random walk $\{T_i\}_{i \geq 0}$ reaches $n - \lceil (1-\zeta)n \rceil$ within the first τ steps:

$$\Pr[\mathcal{F}_{i,j} \text{ occurs for some } i, j \leq \tau] \leq \tau^2 \times 2^{-cn}.$$

Consequently, for, say, $\tau = 2^{cn/3}$, the probability that the random walk reaches $n - \lceil (1-\zeta)n \rceil$ during the first τ steps, is bounded from above by $2^{-cn/3}$. This concludes the proof. ∎

Lemma 6. *There exists a sequence of markets for which Proposition 1, which is stated in the main text, holds. That is, the conditions (i) and (ii) in its statement define a non-empty class (of sequences) of markets.*

Proof. For simplicity of exposition, consider the following two markets, one of odd size $n = 5$ on the left and another of even size $n = 6$ on the right:

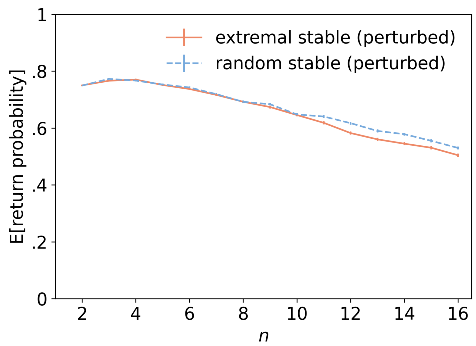
$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} \begin{array}{c} \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & w_5 \end{array} \\ \begin{pmatrix} 4,5 & 3,4 & 2,5 & 5,2 & 1,1 \\ 1,4 & 4,3 & 3,4 & 2,5 & 5,2 \\ 1,3 & 5,2 & 4,3 & 3,4 & 2,5 \\ 1,2 & 2,5 & 5,2 & 4,3 & 3,4 \\ 1,1 & 2,1 & 3,1 & 4,1 & 5,3 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{array} \begin{array}{c} \begin{array}{cccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{array} \\ \begin{pmatrix} 4,6 & 3,5 & 2,6 & 6,2 & 5,3 & 1,1 \\ 1,5 & 4,4 & 3,5 & 2,6 & 6,2 & 5,3 \\ 1,4 & 5,3 & 4,4 & 3,5 & 2,6 & 6,2 \\ 1,3 & 6,2 & 5,3 & 4,4 & 3,5 & 2,6 \\ 1,2 & 2,6 & 6,2 & 5,3 & 4,4 & 3,5 \\ 1,1 & 2,1 & 3,1 & 4,1 & 5,1 & 6,4 \end{pmatrix} \end{array}.$$

It is straightforward to generalize these two instances—by replacing the highest payoff with n, the second-highest payoff with $n-1$, and so forth—to construct a sequence of markets of any size, odd or even; the precise preferences are omitted for brevity.

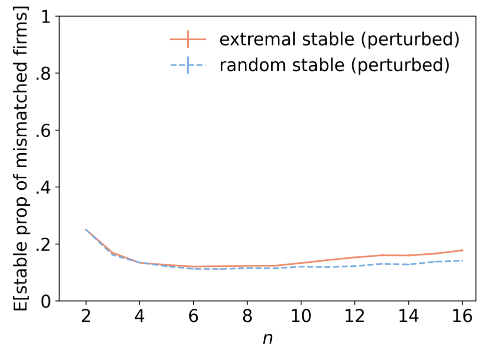
For any market in this sequence, including the markets shown above, there is a unique stable matching $\mu(f_i) = w_i$, for any $i \in [n]$. Furthermore, any firm $f \neq f_n$ is preferred by $\lceil n/2 \rceil - 1$ workers to their stable partners and any worker $w \neq w_1$ is preferred by $\lfloor n/2 \rfloor - 1$ firms to their stable partners. In other words, any agent except the "exclusion" agents f_n and w_1 is preferred by approximately half of agents from the other side to their stable partners. Therefore, Proposition 1 holds for the constructed sequence of markets, as desired. ∎

C. Additional Figures

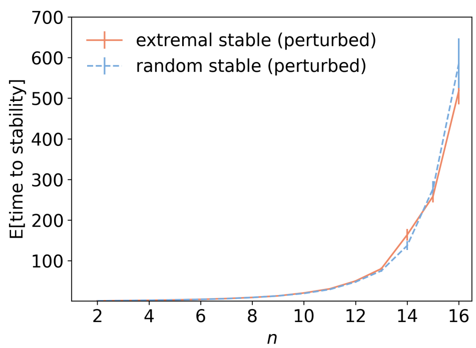
In this section, we present Figures 5-7, analogous to Figures 2-4 in the main text, for the random (best) dynamics where, at each step, a random agent selects her most preferred blocking partner. Specifically, at each step of the dynamics, one agent is chosen uniformly at random from those who have at least one blocking partner. The next matching is then obtained by pairing this chosen agent with her most preferred blocking partner; this corresponds to satisfying the chosen agent's best blocking pair. Similar to the main text, we sample 1,000 random markets for each market size and simulate 300 paths to stability for each minimally perturbed stable matching. As can be seen, all of the insights presented in the main text carry over to these dynamics as well.



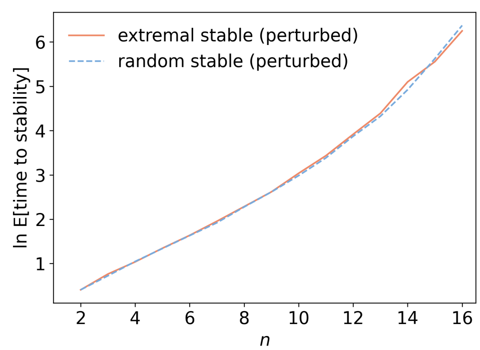
(a) Return probability



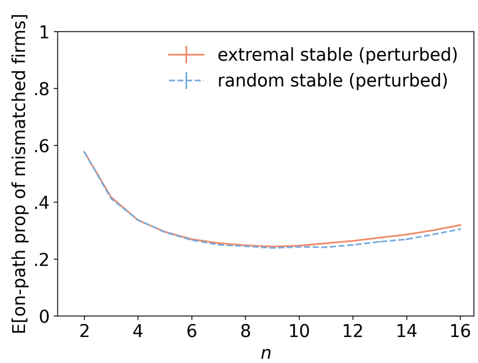
(b) Proportion of mismatched firms/workers in ultimate stable matching



(c) Time to stability

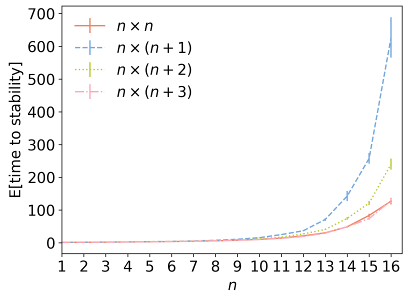


(d) ln [Time to stability]

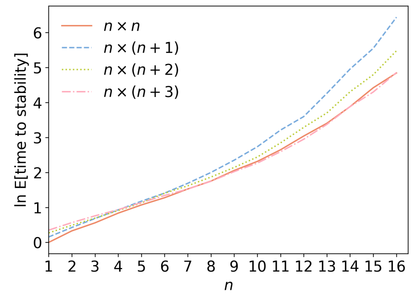


(e) On-path average proportion of mismatched firms/workers

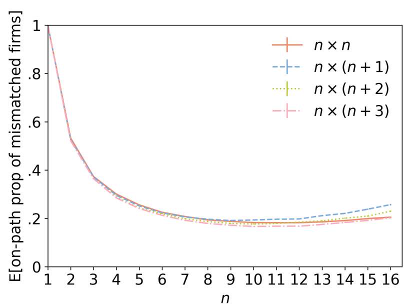
Figure 5: Random $n \times n$ markets with multiple stable matchings



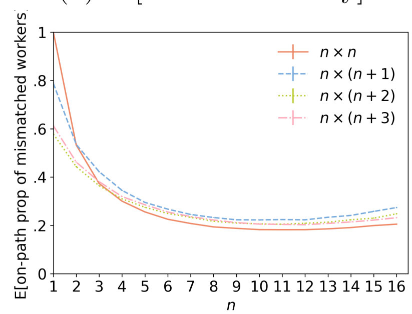
(a) Time to stability



(b) $\ln$ [Time to stability]

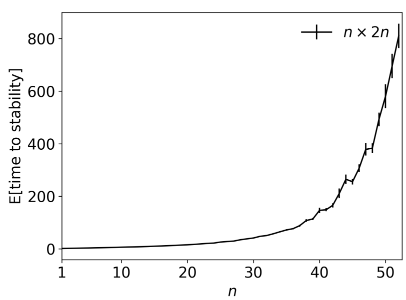


(c) On-path average proportion of mismatched firms

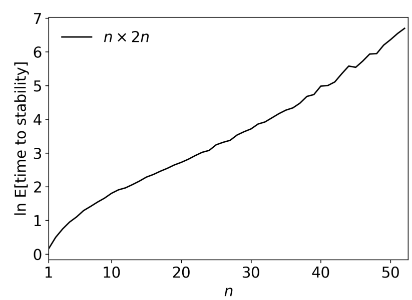


(d) On-path average proportion of mismatched workers

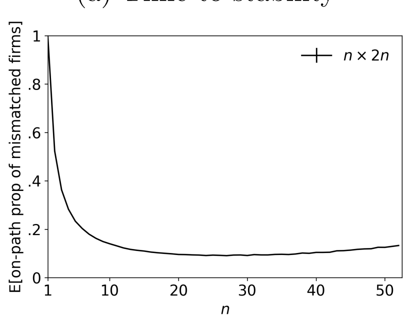
Figure 6: Random $n \times (n+k)$ markets with a unique stable matching, $k \in \{0, 1, 2, 3\}$



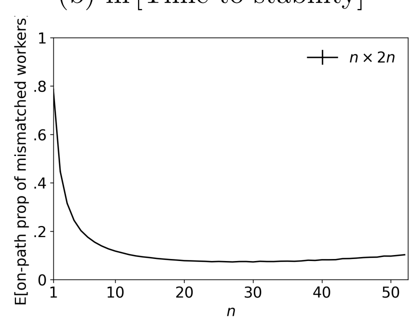
(a) Time to stability



(b) $\ln$ [Time to stability]



(c) On-path average proportion of mismatched firms



(d) On-path average proportion of mismatched workers

Figure 7: Random $n \times 2n$ markets with a unique stable matching

D. Probabilistic Aspects of Fragments

In this section, we focus on random matching markets, of size n, wherein each agent has complete preferences drawn uniformly and independently at random. We first show that in such markets, with high probability, there are no large fragments at all, including trivial ones. Then, we discuss techniques, in the spirit of Knuth (1976)-type integral formulas, that may be potentially useful for the future analysis of fragments in random markets; see Pittel (2019) and references therein for recent applications of similar techniques and their history.

The challenge of estimating the likelihood of fragments is not surprising, given that related problems are known to be complex and remain open. Specifically, Lemma 1 in the main text implies that firms and workers forming a fragment belong to a so-called "closed clique," as defined in the final remark of Pittel, Shepp, and Veklerov (2008): in every stable matching, these agents are matched with each other only. In this remark, Pittel, Shepp, and Veklerov (2008)—who analyze the average number of firm-worker pairs common to all stable matchings—argue that even estimating the probability that some two firms and two workers form such a clique is a hard problem (not yet solved).

Proposition 2. *There are no fragments of sizes $k \geq 0.9n$ with high probability.*

Proof. In what follows, we use an equivalent cardinal representation of ordinal markets with uniformly random preferences. Let $U^f = \{U^f_{ij}\}_{i,j\in[n]}$ and $U^w = \{U^w_{ij}\}_{i,j\in[n]}$ denote the match utilities for firms $\{f_i\}_{i\in[n]}$ and workers $\{w_j\}_{j\in[n]}$, respectively, with all entries drawn randomly and independently from the uniform distribution $U[0,1]$.

We employ a union bound to establish the proposition. There are fewer than n ways to choose $k < n$, a fragment size. Furthermore, for each k, there are at most $\binom{n}{k}^2$ ways to choose subsets $\bar{F}$ and $\bar{W}$ of firms F and workers W, respectively, that can potentially form a fragment of size k. By symmetry, it then suffices to show that for every $k \geq 0.9n$, the probability that firms $\bar{F} = \{f_i\}_{i\in[k]}$ and workers $\bar{W} = \{w_j\}_{j\in[k]}$ constitute a fragment is much smaller than $\binom{n}{k}^{-2} n^{-1}$. To prove this result, we rely on the following two observations.

Observation 1. With probability at least $1 - o\left(\binom{n}{k}^{-2} n^{-1}\right)$, there are at most $\binom{n}{k} n^2$ stable matchings in the market induced by agents $\bar{F}$ and $\bar{W}$.

Let $\mathcal{N}_k$ denote the corresponding number of stable matchings. The observation follows immediately from one-sided Chebyshev's inequality, where $\lambda > 0$,

$$\Pr(X \geq \mathbb{E}\, X + \lambda) \leq \frac{\operatorname{Var} X}{\operatorname{Var} X + \lambda^2},$$

and the following two asymptotic results, obtained in Pittel (1989) and Lennon and Pittel (2009), respectively:

$$\mathbb{E}[\mathcal{N}_k] = (1+o(1))e^{-1}k\ln k,$$
$$\mathbb{E}[\mathcal{N}_k^2] = (1+o(1))\left(e^{-2}+\frac{1}{2}e^{-3}\right)k^2\ln^2 k, \quad k\to\infty.$$

Observation 2. With probability at least $1-o\left(\binom{n}{k}^{-2}n^{-1}\right)$, there are fewer than $0.8n$ pairs of $f_i \in \bar{F}$, $w_j \in \bar{W}$, where $i, j \in [k]$, such that

$$U_{ij}^f > 1-\frac{1}{3\sqrt{n}} \quad \text{and} \quad U_{ij}^w > 1-\frac{1}{3\sqrt{n}}.$$

Let Z be the number of such pairs. In total, there are k^2 possible pairs $f_i \in \bar{F}$, $w_j \in \bar{W}$. Each pair satisfies the above conditions with probability $1/9n$. Thus, by the linearity of expectations, $\mathbb{E}[Z] = k^2/9n$. In addition, $9n/100 \le \mathbb{E}[Z] < n/9$ since $0.9n \le k < n$. Using the multiplicative Chernoff bound, with $\gamma > 0$,

$$\Pr\left[X \ge (1+\gamma)\,\mathbb{E}[X]\right] \le \left[\frac{e^\gamma}{(1+\gamma)^{(1+\gamma)}}\right]^{\mathbb{E}[X]},$$

we obtain, say,

$$\Pr[Z\ge 0.79n] \le \Pr\left[Z \ge (1+\gamma)E[Z]\right]\Big|_{\gamma=6} \le \left[\frac{e^6}{7^7}\right]^{\mathbb{E}[Z]} \le \left[\frac{e^6}{7^7}\right]^{9n/100} \le 1.98^{-n}.$$

In contrast, for $0.9n \le k < n$, using the Stirling approximation, we have

$$\ln\left(\binom{n}{k}^2 n\right) \le \ln\left(\binom{n}{\lfloor 0.9n\rfloor}^2 n\right) = 2\ln\binom{n}{\lfloor 0.9n\rfloor} + \ln n = (1+o(1))2H(0.9)n + \ln n,$$

where $H(p) \equiv -p\ln p - (1-p)\ln(1-p)$. The observation then follows since $e^{2H(0.9)} < 1.93$.

Based on these two observations, with a failure probability of at most $o\left(\binom{n}{k}^{-2}n^{-1}\right)$, the corresponding properties regarding $\mathcal{N}_k$ and Z are both satisfied. Fix any stable matching $\bar{\mu}$ in the market formed by agents $\bar{F}$ and $\bar{W}$; by Observation 1, there are at most $\binom{n}{k}n^2$ such matchings. Due to Observation 2, as $0.9n \le k < n$, this matching contains at least $k - 0.8n \ge 0.1n$ pairs of $f_i \in \bar{F}$, $w_j \in \bar{W}$, with $\bar{\mu}(f_i) = w_j$, such that

$$U_{ij}^f \le 1-\frac{1}{3\sqrt{n}} \quad \text{or} \quad U_{ij}^w \le 1-\frac{1}{3\sqrt{n}}.$$

If $\bar{\mu}$ induces the fragment $(\bar{F}, \bar{W})$, then every agent inside the fragment must prefer her partner under $\bar{\mu}$ to anyone outside the fragment. The corresponding probability is at most

$$\left(1 - \frac{1}{3\sqrt{n}}\right)^{0.1n(n-k)} = e^{-\sqrt{n}(n-k)/30}, \quad n \to \infty.$$

Even after multiplying this probability by at most $\binom{n}{k} n^2$ possibilities to pick $\bar{\mu}$, it still remains $o\left(\binom{n}{k}^{-2} n^{-1}\right)$. This concludes the proof. ∎

Proposition 2 highlights the rarity of large fragments, whether trivial or non-trivial, in random markets. We believe that similar arguments may be used to sharpen the result and show that even (much) smaller fragments are also vanishingly rare.

There are also other tools that can be relevant for examining fragments in random markets. Below, we describe an approach that may be used to obtain integral formulas for certain probabilities, which could be of interest and possibly amenable to an asymptotic analysis. This approach, originating from Knuth (1976) and Pittel (1989), has proved to be useful for analyzing random matching markets; see Pittel (2019) for additional details.

For illustration, consider arbitrary subsets $\bar{F}$ and $\bar{W}$ of firms F and workers W, respectively, each of size $k < n$, and an arbitrary matching $\bar{\mu}$ in the market formed by these subsets. Let $P_k(n)$ denote the probability that $\bar{\mu}$ induces the fragment $(\bar{F}, \bar{W})$; by symmetry, this probability depends only on sizes k and n. Then,

Lemma 7. *We have*

$$P_k(n) = \overbrace{\int \cdots \int}^{2k \text{ times}}_{x,y \in [0,1]^k} \prod_{i,j \in [k], i \neq j} (1 - x_i y_j) \prod_{h \in [k]} (1 - x_h)^{n-k} \prod_{l \in [k]} (1 - y_l)^{n-k} dx dy,$$

where $x = (x_1, x_2, \ldots, x_k)$ *and* $y = (y_1, y_2, \ldots, y_k)$.

Proof. As in the proof of Proposition 2, we employ a cardinal representation of random markets. Let $U^f = \{U^f_{ij}\}_{i,j \in [n]}$ and $U^w = \{U^w_{ij}\}_{i,j \in [n]}$ be the match utilities for firms $\{f_i\}_{i \in [n]}$ and workers $\{w_j\}_{j \in [n]}$, respectively, with all entries selected randomly and independently from the uniform distribution $U[0, 1]$.

Due to symmetry, it suffices to consider $\bar{F} = \{f_i\}_{i \in [k]}$ and $\bar{W} = \{w_j\}_{j \in [k]}$ and the matching $\bar{\mu}$ such that $\bar{\mu}(f_i) = w_i$ for each $i \in [k]$. By definition, $\bar{\mu}$ induces the fragment $(\bar{F}, \bar{W})$ if and only if

$$U_{ii}^f > U_{ij}^f \quad \text{or} \quad U_{jj}^w > U_{ij}^w \qquad \forall i, j \in [k], i \neq j$$
$$U_{ii}^f > U_{ij}^f \qquad \forall i \in [k], j > k,$$
$$U_{jj}^w > U_{ij}^w \qquad \forall j \in [k], i > k.$$

Notably, conditioned on $U_{ii}^f = u_i^f$ and $U_{jj}^w = u_j^w$, with $i, j \in [k]$, the above events are independent. Consequently, the conditional probability that $\bar{\mu}$ induces the fragment $(\bar{F}, \bar{W})$ equals

$$\prod_{i,j\in[k],i\neq j} \left(1-(1-u_i^f)(1-u_j^w)\right) \prod_{h\in[k]} \left(u_h^f\right)^{n-k} \prod_{l\in[k]} \left(u_l^w\right)^{n-k}.$$

By integrating this expression over the cube $[0,1]^{2k}$ and switching to new variables $x_i = 1-u_i^f$ and $y_j = 1 - u_j^w$, $i, j \in [k]$, we obtain the desired integral formula for $P_k(n)$. ∎

The obtained integral formula shares some similarities with those recently analyzed in Pittel (2019). This serves as a promising sign that the described approach may be valuable for analyzing fragments in random markets.

Although large fragments, whether trivial or not, are vanishingly rare, non-triviality appears to play a central role in the rarity of (very) small non-trivial fragments, as suggested by Figure 1 in the main text. Since each firm-worker pair forms a top-top match with probability n^{-2}, and there are n^2 such pairs, top-top match pairs are common in random markets. In fact, it is easy to show that the number of top-top match pairs is asymptotically Poisson with parameter 1. Given that top-top match pairs and their sequences are trivial fragments, this suggests that small trivial fragments are quite common.

In contrast, consider two firms and two workers forming a non-trivial fragment of size $k = 2$, the smallest possible size of a non-trivial fragment. There are roughly n^4 ways to choose such agents and 2 ways to pick a stable matching for them, which can induce the fragment. These agents prefer their stable partners to everyone outside the fragment with a probability of roughly n^{-4}, already matching the number of ways to choose these agents. Non-triviality additionally requires that there is no other stable way to match these agents in the entire market. This additional requirement imposed by non-triviality seems to be the primary reason for the rarity of small non-trivial fragments in random markets.

E. Other Dynamics

In this section, we explore additional classes of reasonable dynamics beyond those discussed in the paper. However, none of these new dynamics even ensure convergence to stability, which is arguably a minimal requirement for viable dynamics.

Proposition 3. *Consider a class of dynamics that, at each step, whenever possible, match multiple (best) blocking pairs.*[2] *No dynamics in this class guarantee convergence to stability.*

Proof. Initialize the following market

$$\begin{array}{c c} & \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array} & \begin{pmatrix} 3,1 & 2,2 & 1,3 \\ 1,3 & 3,1 & 2,2 \\ 2,2 & 1,3 & 3,1 \end{pmatrix} \end{array}$$

at matching $\lambda_1 = (w_1, f_3, f_2)$, specifying the partners of workers w_1, w_2, and w_3, respectively.

It is easy to verify that every given dynamics generate a cycle, and thus fail to attain stability. For illustration, in the first step, there are three (best) blocking pairs: (f_1, w_1), (f_1, w_3), and (f_3, w_1). Consequently, the dynamics necessarily match pairs (f_1, w_3) and (f_3, w_1), leading the market to $\lambda_2 = (f_3, w_2, f_1)$:

$$\lambda_1 \xrightarrow[(f_1,w_3),(f_3,w_1)]{} \lambda_2,$$

where, under the transition arrow, we specify the corresponding blocking pairs.

In fact, the dynamics must follow the deterministic path

$$\lambda_1 \xrightarrow[(f_1,w_3),(f_3,w_1)]{} \lambda_2 \xrightarrow[(f_1,w_2),(f_2,w_1)]{} \lambda_3 = (f_2, f_1, w_3) \xrightarrow[(f_2,w_3),(f_3,w_2)]{} \lambda_4 = \lambda_1,$$

returning to the initial matching λ_1. ∎

In addition,

[2] Formally, at each step i, if the current matching λ_i is unstable, we tabulate the set of all combinations of (best) blocking pairs, consisting of at least two pairs, that can match simultaneously. If such combinations exist, we choose one of them randomly, and the next matching, λ_{i+1}, is obtained by matching all blocking pairs in the chosen combination. Otherwise, if no such combination exists, λ_{i+1} is obtained by satisfying a randomly-chosen (best) blocking pair. Notably, the discussed probabilities can be arbitrary, potentially non-uniform, time-dependent, or even zero.

Proposition 4. *Consider a class of dynamics that, at each step, whenever possible, match a firm-worker pair that are each other's favorite blocking partner.*[3] *No dynamics in this class guarantee convergence to stability.*

Proof. Initialize the market

$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} \begin{array}{c} \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & w_5 \end{array} \\ \begin{pmatrix} 1,3 & 3,1 & 5,4 & 2,5 & 4,1 \\ 1,4 & 2,5 & 3,3 & 5,3 & 4,4 \\ 3,1 & 2,4 & 4,2 & 1,4 & 5,3 \\ 4,2 & 2,3 & 3,5 & 1,1 & 5,2 \\ 2,5 & 4,2 & 5,1 & 3,2 & 1,5 \end{pmatrix} \end{array}$$

at matching $\lambda_1 = (f_5, w_2, f_4, f_2, f_3)$, specifying the partners of workers w_1, w_2, and so on.

Then, every given dynamics generate a cycle, and hence fail to reach stability. Initially, there are three blocking pairs: (f_1, w_2), (f_1, w_4), and (f_5, w_2). In addition, (f_5, w_2) is the only pair of agents that are each other's favorite blocking partner. Therefore, the dynamics must match firm f_5 and worker w_2, leading the market to $\lambda_2 = (w_1, f_5, f_4, f_2, f_3)$:

$$\lambda_1 \xrightarrow[(f_5,w_2)]{} \lambda_2,$$

where under the transition arrow, we specify the satisfied blocking pair.

It is straightforward to check that the dynamics necessarily follow the path

$$\lambda_1 \xrightarrow[(f_5,w_2)]{} \lambda_2 \xrightarrow[(f_1,w_4)]{} \lambda_3 \xrightarrow[(f_2,w_5)]{} \lambda_4 \xrightarrow[(f_4,w_1)]{} \lambda_5 \xrightarrow[(f_1,w_3)]{} \lambda_6 \xrightarrow[(f_3,w_2)]{} \lambda_7 \xrightarrow[(f_2,w_4)]{} \lambda_8 \xrightarrow[(f_5,w_1)]{}$$
$$\xrightarrow[(f_5,w_1)]{} \lambda_9 \xrightarrow[(f_3,w_5)]{} \lambda_{10} \xrightarrow[(f_4,w_3)]{} \lambda_{11} = \lambda_1,$$

returning to the initial matching λ_1. ∎

In what follows, we focus on cardinal markets, with match utilities denoted as $U = \{u^f_{ij}, u^w_{ij}\}_{i,j\in[n]}$. For each pair (f_i, w_j), u^f_{ij} is firm f_i's utility from matching with worker w_j and u^w_{ij} is worker w_j's utility from matching with firm f_i. The sum of these utilities, $u^f_{ij} + u^w_{ij}$, is the total surplus. Without loss of generality, all utilities from being unmatched are normalized to zero. As in the main paper, all preferences are strict, and all worker-firm pairs are mutually acceptable, i.e., $u^f_{ij} > 0$ and $u^w_{ij} > 0$ for all i, j. Then,

[3] Specifically, at each step i, if the current matching λ_i is unstable, we tabulate the set of blocking pairs of agents that are each other's favorite blocking partner. If such blocking pairs exist, we choose one of them randomly, and the next matching, λ_{i+1}, is obtained by satisfying the chosen pair. Otherwise, if no such pair exists, λ_{i+1} is obtained by satisfying a randomly-chosen blocking pair, as usual. The probability that a specific blocking pair is selected to match can be arbitrary.

Proposition 5. *Consider a class of dynamics that, at each step, match a blocking pair having the highest total surplus.*[4] *No dynamics in this class guarantee convergence to stability.*

Proof. Initialize the cardinal market

$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{array} \begin{array}{c} \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{pmatrix} 11,1 & 9,7 & 8,3 & 6,5 \\ 4,4 & 3,6 & 2,8 & 9,3 \\ 1,8 & 2,5 & 8,5 & 3,9 \\ 4,3 & 6,8 & 9,1 & 7,4 \end{pmatrix} \end{array}$$

at matching $\lambda_1 = (w_1, f_4, f_3, f_2)$, which specify the partners of workers w_1, w_2, and so forth.

The dynamics generate a cycle, and thus fail to attain stability. In the first step, there are three blocking pairs: (f_1, w_1) with a total surplus of $12 = 11 + 1$, (f_1, w_4) with a total surplus of $11 = 6 + 5$, and (f_4, w_4) with a total surplus of $11 = 7 + 4$. Consequently, the dynamics match firm f_1 and worker w_1 and lead the market to $\lambda_2 = (f_1, f_4, f_3, f_2)$:

$$\lambda_1 \xrightarrow[(f_1,w_1)]{} \lambda_2.$$

The dynamics then follow the path

$$\lambda_1 \xrightarrow[(f_1,w_1)]{} \lambda_2 \xrightarrow[(f_4,w_4)]{} \lambda_3 \xrightarrow[(f_2,w_3)]{} \lambda_4 \xrightarrow[(f_3,w_4)]{} \lambda_5 \xrightarrow[(f_4,w_2)]{} \lambda_6 \xrightarrow[(f_2,w_1)]{} \lambda_7 \xrightarrow[(f_3,w_3)]{} \lambda_8 \xrightarrow[(f_2,w_4)]{} \lambda_9 = \lambda_1,$$

eventually returning to the initial matching λ_1. ∎

Obviously, the same result can be obtained for dynamics that, at each step, match a blocking pair having the highest *weighted* total surplus; this can be achieved by appropriately scaling the match utilities used in the proof of the above proposition. We also expect similar results for other related dynamics; particularly, for dynamics that, at each step, match a blocking pair having the highest total surplus gain, compared to the previous match.

[4]These dynamics are deterministic at all steps when there is one blocking pair that delivers the highest total surplus. If, at some step, there are multiple such pairs, we choose one of them randomly, and the next matching is obtained by satisfying the chosen pair.

F. Fragility Examples

Example 5. *The example presents a market in which one of the two stable matchings is fragile with respect to arbitrary perturbations.*

Consider the following market with six firms and six workers:

$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{array} \begin{array}{c} \begin{array}{cccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{array} \\ \begin{pmatrix} 6,2 & 4,5 & 3,1 & 2,6 & 5,2 & 1,6 \\ 3,6 & 5,1 & 2,4 & 6,3 & 4,6 & 1,5 \\ 5,5 & 3,3 & 6,2 & 4,1 & 2,4 & 1,3 \\ 1,1 & 4,2 & 5,3 & 6,4 & 3,1 & 2,2 \\ 3,4 & 2,6 & 1,6 & 5,2 & 6,3 & 4,4 \\ 1,3 & 2,4 & 3,5 & 5,5 & 4,5 & 6,1 \end{pmatrix} \end{array}.$$

There are two stable matchings:

$$\mu_F = (f_1, f_2, f_3, f_4, f_5, f_6) \quad \text{and} \quad \mu_W = (f_3, f_1, f_4, f_6, f_2, f_5).$$

Furthermore, each agent has two different stable partners.

By using the Markov structure of the problem with states being matchings, we calculate return probabilities

stable \ unmatch	w_1	w_2	w_3	w_4	w_5	w_6
$\mu_F = (f_1, f_2, f_3, f_4, f_5, f_6)$	0.3108	0.2657	0.2706	0.2769	0.2689	0.2362
$\mu_W = (f_3, f_1, f_4, f_6, f_2, f_5)$	0.9883	0.9810	0.9886	0.9825	0.9801	0.9819

for each almost stable matching, obtained by unmatching worker w_i with his stable partner.

These return probabilities imply that the firm-optimal stable matching μ_F is fragile with respect to arbitrary perturbations. Indeed, each of its almost stable matchings is more likely to converge to the worker-optimal stable matching μ_W. In fact, irrespective of how close we start to μ_F, in order to return back, decentralized interactions need to attain one of its almost stable matchings, and thus are more likely to attain μ_W instead. In that sense, almost stable matchings corresponding to minimal perturbations provide lower bounds on fragility.

In contrast, the worker-optimal stable matching μ_W seems robust. Even though, when perturbed minimally, it can still converge to μ_F, this is very unlikely. Interestingly, robust stable matching μ_W is more egalitarian than fragile stable matching μ_F.[5] △

[5]In our simulations, more egalitarian stable matchings seem to be more robust on average. Nevertheless,

Example 6. *The example presents a market in which all stable matchings are fragile and extremal stable matchings are most fragile.*

Consider the following market with six firms and six workers:

$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{array} \begin{array}{c} \begin{array}{cccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{array} \\ \begin{pmatrix} 6,1 & 5,2 & 4,3 & 3,4 & 2,5 & 1,6 \\ 1,6 & 6,1 & 5,2 & 4,3 & 3,4 & 2,5 \\ 2,5 & 1,6 & 6,1 & 5,2 & 4,3 & 3,4 \\ 3,4 & 2,5 & 1,6 & 6,1 & 5,2 & 4,3 \\ 4,3 & 3,4 & 2,5 & 1,6 & 6,1 & 5,2 \\ 5,2 & 4,3 & 3,4 & 2,5 & 1,6 & 6,1 \end{pmatrix} \end{array}.$$

It has six stable matchings:

$$\mu_1 = \mu_F = (f_1, f_2, f_3, f_4, f_5, f_6), \qquad \mu_4 = (f_4, f_5, f_6, f_1, f_2, f_3),$$
$$\mu_2 = (f_6, f_1, f_2, f_3, f_4, f_5), \qquad \mu_5 = (f_3, f_4, f_5, f_6, f_1, f_2),$$
$$\mu_3 = (f_5, f_6, f_1, f_2, f_3, f_4), \qquad \mu_6 = \mu_W = (f_2, f_3, f_4, f_5, f_6, f_1).$$

Each stable matching corresponds to one of the six "diagonals" in the matrix.

By using the Markov structure, we compute exact values for return probabilities

stable \ unmatch	w_i, any i
$\mu_F = (f_1, f_2, f_3, f_4, f_5, f_6)$	0.2010
$\mu_2 = (f_6, f_1, f_2, f_3, f_4, f_5)$	0.3165
$\mu_3 = (f_5, f_6, f_1, f_2, f_3, f_4)$	0.5035
$\mu_4 = (f_4, f_5, f_6, f_1, f_2, f_3)$	0.5035
$\mu_5 = (f_3, f_4, f_5, f_6, f_1, f_2)$	0.3165
$\mu_W = (f_2, f_3, f_4, f_5, f_6, f_1)$	0.2010

for each almost stable matching. In fact, due to symmetry, it suffices to calculate only three return probabilities.

In this market, every stable matching appears to be fragile. Notably, extremal stable matchings are most fragile. Conversely, more egalitarian stable matchings are less fragile; this observation is similar to Example 5. △

in certain markets, the most egalitarian stable matching may not necessarily be the most robust one; see Boudreau (2011) for a slightly related observation.

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