# Improvable Equilibria

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# Introduction

Communication or intermediation

- precede many interactions: voting, matching, product adoption, etc.
- a possible channel for collusion by auction bidders, market competitors, and the like

**Broad question:** What strategic interactions are susceptible to communication influences or collusion?

# Refined Introduction

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**This project:** When is there potential value in correlation?

**Normal-form game**

$$\Gamma = \Big(N,\ (A_i)_{i\in N},\ (u_i : A \to \mathbb{R})_{i\in N}\Big)$$

- $N = \{1, \ldots, n\}$ is finite set of players
- $A_i$ is a finite set of actions of player $i$
- $A = \prod_{i\in N} A_i$ is the set of action profiles
- $u_i : A \to \mathbb{R}$ is utility of player $i$

# Correlated Equilibria (CE)

**Definition**

A distribution $\mu \in \Delta(A)$ is a correlated equilibrium if

$$\sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})\, u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})\, u_i(a_i', a_{-i})$$

for all $i \in N$ and all $a_i, a_i' \in A_i$

**Interpretation:** $\mu$ generated by a mediator and players best respond by adhering

**Remark:** Nash Equilibria (NE) are CE of the form $\mu = \mu_1 \times \ldots \times \mu_n$

# Approach

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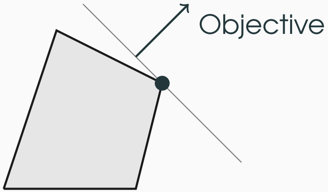
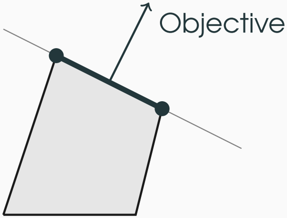
**Our Question:** When is a Nash equilibrium extreme?

# Improvability of non-extreme equilibria

Maximization of a linear objective—e.g., utilitarian welfare—over a polytope $P$:




Two cases:

- If the optimum is unique, it is an extreme point
  - We call objectives with a unique optimum **non-degenerate**
  - Utilitarian welfare is non-degenerate, as we will see
- In knife-edge cases, the whole face of $P$ can be optimal

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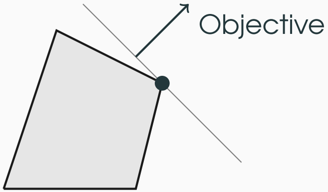


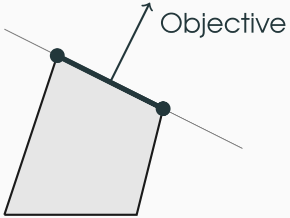


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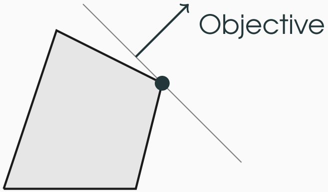


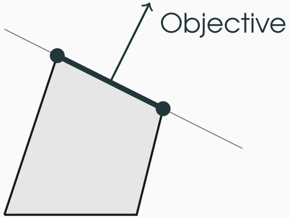


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**Remark:** linear in probabilities, not in actions $\Rightarrow$ a broad class of objectives

# Improvability of non-extreme equilibria 2

**Bauer's Maximum Principle**

Any non-degenerate linear or (quasi-)convex objective attains its maximum at an extreme point

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**Main Insight**

Despite restrictiveness of improvability notion, **many** equilibria are improvable, i.e., **non-extreme**

# Literature Review

- **Value of correlation in 2-player games**: Cripps (1995), Evangelista and Raghavan (1996), Canovas et al. (1999), Nau et al. (2004), Peeters and Potters (1999), Calvó-Armengol (2006), Ashlagi et al. (2008)
- **Communication $\Leftrightarrow$ correlation:** Forges (2020), Bárány (1992), Ben-Porath (1998), Gerardi (2004), Lehrer and Sorin (1997)
- **Communication & collusion in specific contexts**:
  - Bargaining: Crawford (1990), Agranov and Tergiman (2014), Baranski and Kagel (2015)
  - Auctions: McAfee and McMillan (1992), Lopomo et al. (2011), Feldman et al. (2016), Agranov and Yariv (2018), Pavlov (2023)
  - Voting: Gerardi and Yariv (2007), Goeree and Yariv (2011)
  - Matching: Beyhaghi and Tardos (2018), Echenique et al. (2022)
- **Extreme-point approach in info & mech. design:** Kleiner et al. (2021), Arieli et al. (2023), Yang and Zentefis (2024), Kleiner et al. (2024)

# Outline

- **Part 1**
  - Conditions for extremality/improvability
  - Translation to payoffs
  - Applications
- **Part 2**
  - Proof idea
  - Simple description of extreme CE

# Conditions for Extremality

# Extremality of Nash Equilibria

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  $\Rightarrow$ 2-player games not representative

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Genericity can be dropped in any game, by considering **regular** NE only

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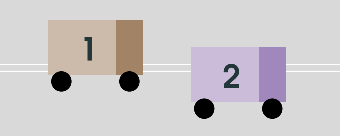
- In a generic game, any NE is regular (Harsanyi, 1973)
- Hence, Theorem 1' $\Rightarrow$ Theorem 1

# Example: 2 Players vs 3 Players

# Example: 2-Player Games

A version of the Game of Chicken by Aumann (1974):




| | Risky | Safe |
|---|---|---|
| Risky | 6, 6 | 10, 7 |
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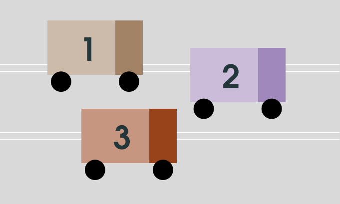
- Aumann (1974): CE can increase utilitarian welfare by shifting weight from (6,6)
- However, the mixed NE is an **extreme point**
- Indeed, it is the optimum for a non-degenerate objective

$$\text{weight of (Risky, Risky) \& (Safe, Safe)} \to \max$$

# Example: 3-Player Games

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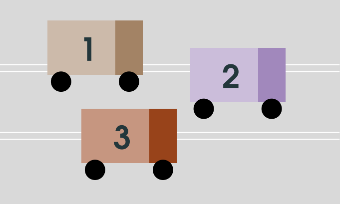


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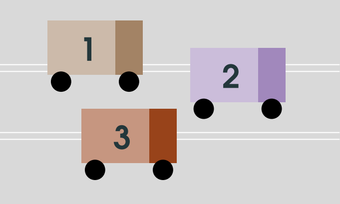


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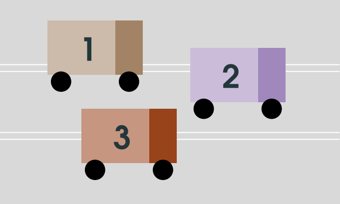


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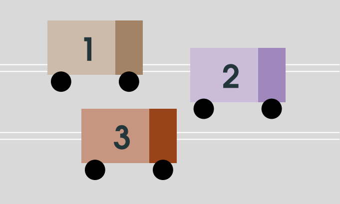


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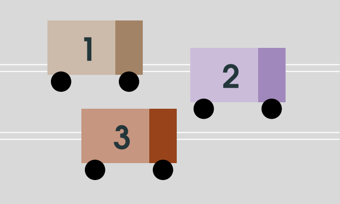


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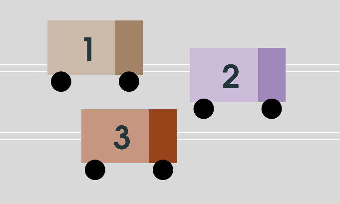


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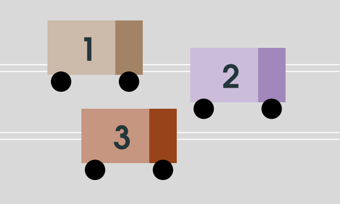


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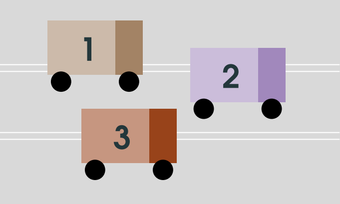


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More than 2 players mixing makes a difference...

# Extreme Points in Payoff Space

# Payoff-extreme Equilibria

- The set of CE $\subset \Delta(A)$ subset of a space of dimension $|A_1| \cdot \ldots \cdot |A_n|$
- Equilibria are often represented via payoffs in $\mathbb{R}^n$

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**Question:** What can we say about payoff-extreme equilibria?

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- Projection of an extreme point **need not** be an extreme point of a projection
- $\Rightarrow$ pure NE and NE with 2 mixers **need not** be payoff-extreme
  - e.g, the mixed NE in the Game of Chicken

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**Proposition**

In a generic game, utilitarian welfare is non-degenerate

# Applications to Particular Classes of Games

# Costly Voting

**Costly voting model of** Palfrey and Rosenthal (1983):

- Two finite groups of voters: $D$ and $R$, $|R| > |D|$
- Voters in $D$ get utility of 1 if $d$-candidate wins and 0 otherwise
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**Other Applications:** games where players want to mismatch actions of others

# Costly Voting

**Costly voting model of** Palfrey and Rosenthal (1983):

- Two finite groups of voters: $D$ and $R$, $|R| > |D|$
- Voters in $D$ get utility of 1 if $d$-candidate wins and 0 otherwise
- Voters in $R$ get utility of 1 if $r$-candidate wins and 0 otherwise
- Majority voting (among those who participate), ties broken randomly
- Costly participation: $c > 0$

Palfrey and Rosenthal (1983): For intermediate values of $c$, all equilibria involve at least one group all mixing

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**Other Applications:** games where players want to mismatch actions of others

- e.g., network games (with substitutes), congestion games, all-pay auctions, Boston matching mechanism

# Symmetric Games

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**Take-away:** caution when focusing on symmetric mixed equilibria in symmetric games

# Games with Unique Correlated Equilibrium

- Games with a unique CE form an open set (Viossat, 2010)
- NE=CE $\Rightarrow$ robustness to incomplete information about payoffs (Einy et al., 2022)

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If a game has a unique correlated equilibrium $\nu$, then $\nu$ is either:

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- No genericity assumption needed thanks to the open-set property

# PART II

# How to Prove Theorem 1

# Proof Idea

**Theorem 1'**

In any game, a regular mixed NE is extreme $\iff$ $\leq 2$ players randomize

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**Idea:** When many players randomize, there are too many ways to correlate their actions, one must be beneficial

Focus on a particular example to illustrate

# How the Proof Goes: Example

- Game with $n$ players, each with 2 actions

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- $2n$ constraints
- Winkler (1988): if $k$ linear constraints are imposed on the set of all distributions $\Delta(A)$, extreme distributions have support $\leq k + 1$
- $\Rightarrow$ support of an extreme CE $\mu$ is bounded by $2n + 1$

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- The same argument applies to equilibria, where players mix over the **same number of pure strategies**
- The main difficulty is handling very asymmetric equilibria

# Key Lemmas for the General Proof

**Support Size of Extreme Correlated Equilibria (follows from Winkler (1988))**

If $\mu$ is an extreme correlated equilibrium, then

$$\text{supp}(\mu) \leq 1 + \sum_{i \in N} |A_i| \cdot (|A_i| - 1)$$

**Support Size of Regular Nash Equilibria (McKelvey and McLennan, 1997)**

For a regular Nash equilibrium, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$:

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Let's combine these two observations

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By the lemmas from the previous slide:

$$\prod_{i=1}^{n} |A_i| \leq 1 + \sum_{i \in N} |A_i| \cdot (|A_i| - 1) \qquad \Longleftarrow \text{the bound on the support of extreme CE}$$

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- The proposition is proved via majorization & Schur convexity

# What Extreme CE Look Like

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**Question:** What is the structure of extreme CE?

- For general games, we only know that extreme CE have small support
- For symmetric games and symmetric CE, we can say more

# Symmetric CE and Exchangability

**Observation:**

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**Theorem (de Finetti)**

Any infinite exchangeable sequence $a_1, a_2, a_3 \ldots$ is a mixture of i.i.d. distributions

# Extreme Symmetric CE with Many Players

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- A version of Proposition 2 holds: sampling without replacement instead of i.i.d.

▸ details

# Conclusions

Several papers effectively show extremality of NE in specific contexts:

- Tullock contests, Cournot and Bertrand, patent races, location games (Einy, Haimanko, and Lagziel, 2022)
- First-price auctions (Feldman, Lucier, and Nisan, 2016)
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Thank you!

# General linear objectives

- Consider a NE $\nu$
- For simplicity, $\nu$ has full support
- By Farkas lemma, a linear objective $L$ can be improved for $\nu \iff L$ **cannot** be expressed as

$$L(\mu) = C + \sum_{i, a_i, a_i', a_{-i}} \mu(a) \cdot \lambda_i(a_i, a_i') \cdot \left(u_i(a_i, a_{-i}) - u_i(a_i', a_{-i})\right)$$

  for some $\lambda_i(a_i, a_i') \geq 0$.
- For non-extreme NE $\nu$, "bad" $L$ form a lower-dimensional subspace

back

# Extreme Symmetric CE with Any Number of Players

Consider $n$ players with $m$ actions each

**Proposition**

Any extreme symmetric CE can be obtained as follows:

- there are $M$ urns, each with $n$ balls labeled by actions

$$1 \leq M \leq m(m-1)+1$$

- an urn is selected at random according to $p \in \Delta_M$, secretly from players
- players draw balls sequentially without replacement
- $i$'s action = her ball's label, no incentive to deviate

**Remark:** If $n$ is large, sampling without replacement can be approximated by i.i.d.

▸ back

# Bayesian games

**Bayesian game**

$$\mathcal{B} = \Big(N,\ (A_i)_{i\in N},\ (T_i)_{i\in N},\ \tau \in \Delta(T),\ (u_i : A \times T_i \to \mathbb{R})_{i\in N}\Big)$$

- Each player $i \in N$ has a type $t_i \in T_i$
- Profile of types $(t_1, \ldots, t_n) \in T$ sampled from $\tau$
- Each player $i$ observes her realized type
- Utility $u_i : A \times T_i \to \mathbb{R}$ depends on the action profile and $i$'s type

**Technical assumption:** sets of types $T_i$ are finite

# Bayesian Correlated Equilibria (BCE)

**Definition**

A joint distribution $\mu \in \Delta(A \times T)$ is a Bayesian correlated equilibrium if

- The marginal on $T$ coincides with $\tau$
- For each player $i$, type $t_i$, recommended action $a_i$, and deviation $a_i'$,

$$\sum_{(a_{-i},t_{-i})} \mu((a_i,t_i),(a_{-i},t_{-i}))\, u_i(a_i,t_i,a_{-i}) \geq \sum_{(a_{-i},t_{-i})} \mu((a_i,t_i),(a_{-i},t_{-i}))\, u_i(a_i',t_i,a_{-i})$$

**Interpretation:** a mediator having access to realized types recommends actions to each player. Two aspects:

1. **Ex-ante coordination:** a source of correlated randomness (as in CE)
2. **Information sharing:** providing $i$ more info about $t_{-i}$ than contained in $t_i$

**Remark:** Bergemann and Morris (2016) allow for a broader class of BCE, where player $i$ observes a noisy signal about her type

# Induced Complete Information Game

We can associate a complete information normal form game $\Gamma_{\mathcal{B}}$ with $\mathcal{B}$:

- Replace $A_i$ with set of functions $\sigma_i : T_i \to A_i$
- $\Sigma_i$ is the set of all such $\sigma_i$
- Utility $v_i : \Sigma \to \mathbb{R}$ is given by

$$v_i(\sigma) = \sum_{t \in T} \tau(t) \cdot u_i\big((\sigma_1(t_1), \ldots, \sigma_n(t_n)),\ t_i\big)$$

**Induced Complete Information Game**

$$\Gamma_{\mathcal{B}} = \big(N,\ (\Sigma_i)_{i \in N},\ (v_i)_{i \in N}\big)$$

**Question:** What is a relation between CE of $\Gamma_{\mathcal{B}}$ and BCE of $\mathcal{B}$?

# Induced complete information game

**Relationship between equilibria in $\Gamma_{\mathcal{B}}$ and $\mathcal{B}$**

CE in $\Gamma_{\mathcal{B}}$ $\Leftrightarrow$ ex-ante coordination in $\mathcal{B}$ with no information sharing

- i.e., BCE such that $a_i$ is independent of $t_{-i}$ conditionally on $t_i$

Nash in $\Gamma_{\mathcal{B}}$ $\Leftrightarrow$ Bayes-Nash in $\mathcal{B}$

**Observation:** Generic $\mathcal{B}$ leads to generic $\Gamma_{\mathcal{B}}$

- $\Rightarrow$ we can apply our theorem to $\Gamma_{\mathcal{B}}$ to learn about generic $\mathcal{B}$

**Corollary**

For a generic Bayesian game, a Bayes-Nash equilibrium is improvable via ex-ante coordination $\Longleftrightarrow$ at least 3 players randomize

Applies to Bayesian games where players randomize in equilibrium, e.g., costly voting with private types (Feddersen and Pesendorfer, 1997) and contests (Baranski and Goel, 2024)

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